Paper:

Regulation on Nash Equilibriums in Game-Based Control Systems

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The game-based control system (GBCS), which is a cross between control theory and game theory, was established to investigate objects driven by the external input and their own interests. This paper studies a special type of GBCSs with rational players, open-loop Nash equilibrium of which is unique under any given initial state and macro-regulation. To make low-level micro-followers "better" by resorting to macro-regulation, two kinds of regulation on Nash Equilibriums are discussed under Pareto and Kaldor-Hicks criteria, respectively. By resorting to macro-regulation, one occurs the Pareto improvement on Nash Equilibriums, the other achieves the potential Pareto improvement while generating the optimal action profile, both of which reduce the inconsistency between individual and collective rationality. Some conditions are given to determine the solvability of corresponding regulation problems on Nash Equilibriums.

Keywords: Nash equilibrium, game-based control system, optimal control, differential game

1. Introduction

In 2005, to drive fundamental scientific researches, *Science* listed a series of most important questions of the 21st century [9], in which the evolution of cooperative behaviors was contained [16]. However, as the other side of the coin, the noncooperation is always with the cooperation, both of which are ubiquitous. In non-cooperative game theory involving two or more players, the Nash equilibrium is a stable strategy profile. However, from a holistic perspective, it is possible that Nash equilibrium is not optimal, because of the inconsistency between individual and collective rationality. In prisoner's dilemma the Nash equilibrium is mutual defection, the self-interested strategy profile, rather than cooperation that is Pareto efficient [7, 13].

It have been working on how to achieve the consistency between individual and collective rationality for years, for example, repeated games [2, 11, 15, 18]. An another widely accepted and effective mechanism is to introduce a third party [5, 6], for example, the government closing some roads in Braess' paradox to achieve the improvement on Nash Equilibriums. In social or economic even biological systems, hierarchical structures like that in Braess' paradox can be found everywhere, such as, the network security [1], the smart grid [17]. A hierarchical structure consists of two levels at least, in which the decision sequence and the asymmetric information are contained. Roughly speaking, the players in the low-level are followers, following the decision preannounced by the high-level regulator or leader, which can be regarded as the third party. In fact, the third party serve primary players and there are at least two kinds of principle for the third party to govern. One is the Pareto efficiency, the other is the Kaldor-Hicks efficiency. The former aims to reach a the Pareto optimal (or efficient) strategy profile, at which no alternative strategy profile that would make some people better off without making anyone worse off. The latter pursues minimum costs or maximum profits of the whole, which is the potential Pareto optimum. Because redistributing suitably the costs or profits of the whole under the Kaldor-Hicks criterion is able to reduce everyone's costs or improve everyone's profits, which is a Pareto improvement.

The game theory established by J. Neumann and O. Morgenstern is static [14], then it is generalized to the differential games, which is dynamic [3, 8]. Differential games have intimate connection with the control theory, especially, optimal control [3]. In the traditional control theory, control objects usually are machines, which are driven by physical laws and passively accept control input. However, when objects are agents with intelligence, they are not always passive to be controlled, since it is possible that objects' interests have an effect on their behaviors.

To apply the control theory into systems with intelligent agents, it is necessary to introduce the game theory into it. Game-based control systems(GBCSs), which were first attempted to discuss in [10, 12], were established recently by crossing the differential game theory and the control theory [20]. GBCSs are hierarchical, however, compared with Stackelberg games, the leader or the macro-regulator in the high-level of GBCSs has an intention, which could be minimum own payoff, controllability, observability, and so on. In other words, the regulator does not participate in but control the low-level game containing low-level micro-followers, which makes GBCSs more general than Stackelberg games. Until now, controllability of GBCSs has been investigated widely. Controllability on states was discussed in [20], then these obtained achievements are generalized to the controllability on macro-states [22] and stochastic GBCSs [21, 23]. Observability and controllability on Nash Equilibriums of GBCSs are discussed, under which Nash Equilibriums of microcosmic agents can be identified by the macro-regulator via outputs and Nash Equilibriums can be control from one to another, respectively [19].

For GBCSs, followers' actions constitute Nash Equilibriums under the given initial state and the decision preannounced by the regulator. However, it is possible that the Nash equilibrium derived from the initial state is not Pareto or Kaldor-Hicks efficient, so it is necessary to regulate and improve Nash Equilibriums. In this paper, we investigate GBCSs with individually rational players. Under the management of the regulator, low-level micro-followers can reach to a new Nash equilibrium, which is a Pareto improvement or Kaldor-Hicks efficient compared with the Nash equilibrium generating by the original linear quadratic differential game without the third party. The main contributions of this paper are as follows. Under the Pareto and Kaldor-Hicks criterion, this paper gives some necessary and sufficient conditions to reduce the inconsistency between individual and collective rationality by macro-regulation.

The remainder of this paper is organized as follows. Section 2 introduces the GBCS and some related concepts and necessary assumptions. Section 3 contains our main results, following by a brief conclusion in section 4.

2. Game-Based Control Systems

In this part, we give some necessary preliminaries, which will be used later. First, some notations are listed. The transposition, the rank and the image (column) space of matrix M are denoted as $M^\top$, $\mathrm{Rank}(M)$, $\mathrm{Im}(M)$, respectively. Matrix $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix, and $\mathbf{0}_{n\times m} \in \mathbb{R}^{n\times m}$ ($\mathbf{0}_n \in \mathbb{R}^n$) is zero matrix (zero column vector), subscripts of which will be omitted if no confusion is caused.

A GBCS with one macro-regulator and two micro-followers can be described as follows [20]:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1u_1(t) + B_2u_2(t) \quad . \; . \quad (1)$$

$$J_i = \frac{1}{2}x^\top(T)Q_{iT}x(T) + \frac{1}{2}\int_0^T u_i(t)^\top R_{ii}u_i(t)\mathrm{d}t, \quad (2)$$

where $x(t) = [x^0(t)^\top, x^1(t)^\top, x^2(t)^\top]^\top \in \mathbb{R}^n$, matrices Q_{iT}, $i = 1,2$, are symmetric and R_{ii}, $i = 1,2$, are positive definite. In this system, $x^0(t) \in \mathbb{R}^{n_0}$ and $u(t) \in \mathbb{R}^m$ are respectively the macro-state and the macro-regulation (control, decision or action) of the macro-regulator; $x^i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are respectively the $i-$th agent's state and action. Moreover, matrices in (1)-(2) have appropriate dimensions and are known to the macro-regulator.

Next, we introduce the sequence of decision-making and the information structure of (1)-(2). The macro-regulator in the high-level announces its action $u(t), t \in [0,T]$ first, then agents in the low-level simultaneously adopt action $u_i(t), t \in [0,T]$ to minimize the cost function $J_i(\cdot)$, $i = 1,2$, respectively. And the definition on the Nash equilibrium is given as follows.

Definition 1: [3] Under given u and x_0, (u_1^*, u_2^*) is said to be a Nash equilibrium of (1)-(2), if $J_i(x_0, u, u_i^*, u_{-i}^*) \le J_i(x_0, u, u_i, u_{-i}^*)$, $i = 1,2$, where u_{-i}^* is the opponent's action of player i.

From the above definition, it can be seen that any player unilaterally breaking away from the Nash equilibrium does not decrease its own payoffs. Therefore, the rational player have no motivation to break away from the Nash equilibrium. In this paper, Nash Equilibriums to be discussed are open-loop [3]. That is to say, the information that player i obtains in GBCS (1)-(2) is just the initial state $x(0)$ besides the model structure and the macro-regulation. To make the solution of (1) exist uniquely, admissible actions $u(t), u_i(t)$ are square integrable functions on $[0,T]$, which constitute corresponding admissible action sets.

For the $i-$th player ,we construct a Hamiltonion H_i as follows:

$$H_i = x^\top Q_i x + u_i^\top R_{ii} u_i + \lambda_i^\top (Ax + Bu + \sum_{j=1}^{2} B_j u_j).$$

Suppose that (u_1^*, u_2^*) is the Nash equilibrium of (1)-(2) under $u(t)$ and $x(0)$. Using the maximum principle, we have

$$\begin{cases} \dot{z}(t) = \bar{A}z(t) + \bar{B}u(t), \\ u_1^* = -R_{11}^{-1}B_1^\top \lambda_1(t), \\ u_2^* = -R_{22}^{-1}B_2^\top \lambda_2(t), \\ x(0) = x_0, \\ \lambda(T) = Q_T x(T) \end{cases} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where $z(t) = [x(t)^\top, \lambda_1(t)^\top, \lambda_2(t)^\top]^\top$, $\bar{B} = [B^\top, \mathbf{0}_{2n\times m}]^\top$, $S = [S_1, S_2]$, $S_i = B_i R_{ii}^{-1} B_i^\top$, $P = -I_2 \otimes A^\top$ and

$$\bar{A} = \begin{bmatrix} A & -S \\ \mathbf{0} & P \end{bmatrix}.$$

When $B = \mathbf{0}_{n\times m}$ or $u(t) \equiv \mathbf{0}_m$, literature [4] investigated Nash Equilibriums in (1)-(2) on the basis of the following assumption.

Assumption 1: [4] Riccati differential equation

$$\begin{cases} \dot{K}_i = -A^\top K_i - K_i A + K_i S_i K_i, \\ K_i(T) = Q_{iT} \end{cases} \quad . \; . \; . \; . \; . \quad (4)$$

has a symmetric solution $K_i(t)$ on $[0,T]$, $i = 1,2$.

When matrices $Q_{iT}, i = 1,2$, are positive semidefinite, Riccati differential equation (4) always has a unique solution, which is symmetric [4]. Moreover, an another assumption will be used in Section 3, which is listed as follows.

Assumption 2: Riccati differential equation

$$\begin{cases} \dot{K} = -A^\top K - KA + KS^\Sigma K, \\ K(T) = Q_T^\Sigma \end{cases} \quad . \; . \; . \; . \; . \; . \quad (5)$$

has a symmetric solution $K(t)$ on $[0,T]$, where $S^{\Sigma} = S_1 + S_2$ and $Q_T^{\Sigma} = Q_{1T} + Q_{2T}$.

In this paper, we discuss the regulation of Nash Equilibriums in GBCS (1)-(2) under the setting that (1)-(2) has a unique Nash equilibrium for any given initial state and admissible macro-regulation. Then we give a result on the uniqueness of Nash equilibrium based on the two-point boundary-value problem (3).

Lemma 1: [19] Under Assumption 1, the following three statements are equivalent.

1) For any given $u(t)$ and $x(0)$, GBCS (1)-(2) has a unique Nash equilibrium.

2) There exist $u(t)$ and $x(0)$, such that GBCS (1)-(2) has a unique Nash equilibrium.

3) Determinant $|H_T| \neq 0$, where $Q_T = [Q_{1T}^\top, Q_{2T}^\top]^\top$ and

$$H_T = [-Q_T, I_{2n}]e^{\bar{A}T}[\mathbf{0}_{2n\times n}, I_{2n}]^\top. \quad \ldots\ldots \quad (6)$$

Partitioning matrices $e^{\bar{A}t}$ and $e^{-\bar{A}t}$ into some blocks gets

$$e^{\bar{A}t} = \begin{bmatrix} \phi_{11}^t & \phi_{12}^t \\ \phi_{21}^t & \phi_{22}^t \end{bmatrix}, e^{-\bar{A}t} = \begin{bmatrix} \varphi_{11}^{-t} & \varphi_{12}^{-t} \\ \varphi_{21}^{-t} & \varphi_{22}^{-t} \end{bmatrix},$$

where $\phi_{11}^t, \varphi_{11}^{-t} \in \mathbb{R}^{n\times n}$. Obviously, $\phi_{21}^t \equiv \varphi_{21}^{-t} \equiv \mathbf{0}$, $t \in [0,T]$. Therefore it can be found that under any given $u(t)$ and $x(0)$,

$$\begin{cases} x(t) = \phi_{11}^t(x(0) + \int_0^t \varphi_{11}^{-\tau}Bu(t)\mathrm{d}\tau) + \phi_{12}^t\lambda(0), \\ \lambda(t) = \phi_{22}^t\lambda(0). \end{cases} \quad (7)$$

On the basis of $\lambda(T) = Q_T x(T)$, we obtain

$$H_T\lambda(0) = J_T x(0) + J_T\int_0^T \varphi_{11}^{-t}Bu(t)\mathrm{d}t, \quad \ldots \quad (8)$$

where $H_T = \phi_{22}^T - Q_T\phi_{12}^T$, $J_T = Q_T\phi_{11}^T$. From (8), it can be found that

$$\lambda(0) = H_J x(0) + H_J\int_0^T \varphi_{11}^{-t}Bu(t)\mathrm{d}t, \quad \ldots \quad (9)$$

where $H_J = H_T^{-1}J_T$.

Substituting (9) into (7) obtains the trajectory and the equilibrium of (1)-(2) as follows:

$$\begin{cases} x(t) = W_x^t(x(0) + \int_0^T \varphi_{11}^{-t}Bu(t)\mathrm{d}t) \\ -\phi_{11}^t\int_t^T \varphi_{11}^{-t}Bu(t)\mathrm{d}t, \\ \lambda(t) = W_\lambda^t(x(0) + \int_0^T \varphi_{11}^{-t}Bu(t)\mathrm{d}t) \\ R_b\lambda(t) = [u_1^*(t)^\top, u_1^*(t)^\top]^\top, \end{cases} \quad \ldots \quad (10)$$

where $W_x^t = \phi_{11}^t + \phi_{12}^t H_J$, $W_\lambda^t = \phi_{22}^t H_J$,

$$R_b = -\begin{bmatrix} R_{11}^{-1}B_1^\top & \mathbf{0} \\ \mathbf{0} & R_{22}^{-1}B_2^\top \end{bmatrix}. \quad \ldots\ldots \quad (11)$$

Under Assumption 1 and $|H_T| \neq 0$, the Nash equilibrium of (1)-(2) is uniquely determined by corresponding initial state $x(0) = x_0$, and control $u(t)$ of the high level, $t \in [0,T]$. Denote $N_e(x_0, u(t))$ the Nash equilibrium determined by x_0 and $u(t)$ in (1)-(2).

3. Main Results

In this section, we discuss the regulation on Nash Equilibriums.

First, we consider the following differential game

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{2} B_j u_j(t) \quad \ldots\ldots \quad (12)$$

with cost functions J_i in (2). Under Assumption 1 and $|H_T| \neq 0$, the Nash equilibrium of (12)-(2) is unique. According to (10), the Nash equilibrium of (12)-(2) satisfies that

$$\begin{cases} u_1^*(t) = -R_{11}^{-1}B_1^\top e^{-A^\top t}H_J^1 x(0), \\ u_2^*(t) = -R_{22}^{-1}B_2^\top e^{-A^\top t}H_J^2 x(0), \end{cases} \quad \ldots \quad (13)$$

where $H_J^1 = [I_n, \mathbf{0}]H_J$ and $H_J^2 = [\mathbf{0}, I_n]H_J$. Denote $N_e(x_0)$ the Nash equilibrium determined by $x(0) = x_0$ in (12)-(2).

By adding the term $Bu(t)$ on macro-regulation in (12), the third party is introduced. Under this, will the new Nash equilibrium (10) be better for every initial state? There are two evaluation criteria at least. One is Pareto efficiency, the other is Kaldor-Hicks efficiency. The former aims to reach a the Pareto optimal (or efficient) strategy profile, at which no alternative strategy profile that would make some people better off without making anyone worse off. The latter pursues minimum costs or maximum profits of the whole, which is the potential Pareto optimum. Denote $J_i(x_0, u(t), N_e(x_0, u(t)))$ (or $J_i(x_0, u(t))$) and $J_i(x_0, N_e(x_0))$ (or $J_i(x_0)$) the cost of the player i in (1)-(2) and (12)-(2), respectively.

Based on (10), it can be seen that for any given initial state $x(0)$ and high-level regulation $u(t)$, (1)-(2) satisfies

$$x(T) = W_x^T(x(0) + \int_0^T e^{-At}Bu(t)\mathrm{d}t). \quad \ldots \quad (14)$$

Thus the cost function $J_i(x_0, u(t), N_e(x_0, u(t)))$ of (1)-(2) can be simplified into

$$(x(0) + \int_0^T e^{-At}Bu(t)\mathrm{d}t)^\top Q^i(x(0) + \int_0^T e^{-At}Bu(t)\mathrm{d}t)$$

where $Q_u^i = (H_J^i)^\top \int_0^T e^{-At}B_i R_{ii}^{-1}B_i^\top e^{-A^\top t}\mathrm{d}t H_J^i$, $Q_x^i = (W_x^T)^\top Q_{iT}W_x^T$, $Q^i = Q_u^i + Q_x^i$, $i = 1,2$. Specially, the cost function $J_i(x_0, N_e(x_0))$ of (12)-(2) can be reduced into

$$x(0)^\top Q^i x(0).$$

3.1. Pareto Criterion

We define the regulation on Nash Equilibriums as follows.

Definition 2: The regulation problem on Nash Equilibriums of (1)-(2) is solvable for $x(0) = x_0$ under the Pareto criterion, if there exists $u(t)$, such that $J_i(x_0, u(t), N_e(x_0, u(t))) \leq J_i(x_0, N_e(x_0))$, $i = 1,2$, with at least one of the inequalities being strict.

The regulation problem is said to be globally solvable under the Pareto criterion, if the regulation problem is solvable for any initial state x_0 under the Pareto criterion.

Based on Definition 2, we can say that the regulation problem on Nash Equilibriums of (1)-(2) is not globally solvable under the Pareto criterion, if there exists initial state x_0, for any admissible $u(t)$, satisfying one of following three conditions:

1) $J_i(x_0,u(t),N_e(x_0,u(t))) = J_i(x_0,N_e(x_0))$, $i=1,2$,
2) $J_1(x_0,u(t),N_e(x_0,u(t))) > J_1(x_0,N_e(x_0))$,
3) $J_2(x_0,u(t),N_e(x_0,u(t))) > J_2(x_0,N_e(x_0))$.

Now we claim that under the Pareto criterion, the regulation problem on Nash Equilibriums of (1)-(2) is not globally solvable when Q^i, $i=1,2$ are positive semidefinite. When initial state $x(0)=\mathbf{0}$, results $x(T)=\mathbf{0}$ and $N_e(x_0)\equiv\{\mathbf{0},\mathbf{0}\}$ can be implied from (14) and (13), respectively. Thus, in this case, the regulation problem is not globally solvable, since $J_i(x_0,u(t),N_e(x_0,u(t)))\geq 0 = J_i(x_0,N_e(x_0))$ satisfies the above conditions, if Q^i, $i=1,2$, are positive semidefinite. Namely, if the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable under the Pareto criterion, then Q^i has at least one negative eigenvalue.

Proposition 1: Under Assumption 1 and $|H_T|\neq 0$, the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable under the Pareto criterion, if and only if it is solvable for $x(0)=\mathbf{0}$.

Proof: It suffices to show the sufficiency, since the necessity is trivial. If the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable for $x(0)=\mathbf{0}$ under the Pareto criterion, then based on the Lemma 3 in [22] that

$$\{\int_0^T e^{-At}Bu(t)\mathrm{d}t\,|u(t) \textit{ is admissible}\} = \mathrm{Im}(W_T^C), \tag{15}$$

there exists $\bar{y}\in\mathbb{R}^n$, such that

$$\begin{cases} (W_T^C\bar{y})^\top Q(W_T^C\bar{y}) < 0, \\ (W_T^C\bar{y})^\top Q^i(W_T^C\bar{y}) \leq 0, i=1,2. \end{cases}, \tag{16}$$

where $W_T^C=\int_0^T e^{-At}BB^\top(e^{-At})^\top\mathrm{d}t$, $Q=Q^1+Q^2$. Furthermore, if $(W_T^C\bar{y})^\top Q^i(W_T^C\bar{y})<0$, then for any given $x(0)$, there exists $K_i=|\frac{2\cdot x^\top Q^i W_T^C\bar{y}}{\bar{y}^\top W_T^C Q^i W_T^C\bar{y}}|\in\mathbb{R}$, satisfying for any $k\in\mathbb{R}$ with $|k|>K_i$,

$$\begin{array}{rl} & (x+kW_T^C\bar{y})^\top Q^i(x+kW_T^C\bar{y}) \\ = & k^2\cdot(W_T^C\bar{y})^\top Q^i(W_T^C\bar{y}) + 2k\cdot x^\top Q^i W_T^C\bar{y} + x^\top Q^i x \\ < & x^\top Q^i x, \end{array}$$

since $k^2\cdot(W_T^C\bar{y})^\top Q^i(W_T^C\bar{y})+2k\cdot x^\top Q^iW_T^C\bar{y}$ is a quadratic function on k. Otherwise, if $(W_T^C\bar{y})^\top Q^i(W_T^C\bar{y})=0$, then for any given $x(0)$, for any k satisfying $x^\top Q^i W_T^C\bar{y}\cdot k\leq 0$, we have

$$\begin{array}{rl} & (x+kW_T^C\bar{y})^\top Q^i(x+kW_T^C\bar{y}) \\ = & 2k\cdot x^\top Q^i W_T^C\bar{y} + x^\top Q^i x \\ \leq & x^\top Q^i x. \end{array}$$

Thus, if $(W_T^C\bar{y})^\top Q^i(W_T^C\bar{y})<0, i=1,2$, then we set $K=\max\{K_1,K_2\}+1$ and $y=K\bar{y}$, under which

$$\begin{cases} (x+W_T^C y)^\top Q(x+W_T^C y) < x^\top Q x, \\ (x+W_T^C y)^\top Q^i(x+W_T^C y) \leq x^\top Q^i x, i=1,2, \end{cases} \tag{17}$$

holds. Otherwise, without loss of generality, suppose that $(W_T^C\bar{y})^\top Q^1(W_T^C\bar{y})<0$ and $(W_T^C\bar{y})^\top Q^2(W_T^C\bar{y})=0$, then suitable K can be constructed as follows:

$$K=\begin{cases} K_1+1, x^\top Q^2 W_T^C\bar{y}\leq 0 \\ -K_1-1, x^\top Q^2 W_T^C\bar{y}>0. \end{cases}$$

When $y=K\bar{y}$, (17) holds as well. Combining Definition 2 and Lemma 3 in [22] with simplified cost functions obtains that the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable under the Pareto criterion. ■

3.2. Kaldor-Hicks Criterion

It is possible that Nash equilibrium is not optimal from the holistic perspective, because of the inconsistency between individual and collective rationality. Next, we discuss the regulation problem on Nash Equilibriums of (1)-(2) under the Kaldor-Hicks criterion. Under the Kaldor-Hicks criterion, the optimal action profile satisfies minimum cost function

$$J := J_1 + J_2. \tag{18}$$

Under the Kaldor-Hicks criterion, u_1 and u_2 can be regarded as an entirety. In this sense, determining the optimal action profile of (1)-(18) can be converted into solving a problem of linear quardratic optimal control. According to the maximum principle, we can get equations defining the dynamics of the agents' optimal action profile $O_{kh}(x_0,u(t)) := (\hat{u}_1^*,\hat{u}_2^*)$ as follows:

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{u}_1^*(t) = -R_{11}^{-1}B_1^\top\hat{\lambda}(t), \\ \hat{u}_2^*(t) = -R_{22}^{-1}B_2^\top\hat{\lambda}(t), \\ x(0) = x_0, \\ \hat{\lambda}(T) = Q_T^\Sigma x(T), \end{cases} \tag{19}$$

where $\hat{x}(t)=[x(t)^\top,\hat{\lambda}^\top(t)]^\top$, $\hat{B}=[B^\top,\mathbf{0}]^\top$ and

$$\hat{A}=\left[\begin{array}{cc} A & -S^\Sigma \\ \mathbf{0} & -A^\top \end{array}\right].$$

It is well known that linear quardratic problem (1)-(18) has a (unique) solution for all $x_0\in\mathbb{R}^n$ if and only if Assumption 2 holds [4].

Inspired by (14) it is necessary to express the optimal action profile of (1)-(18) as a time-variant function on $x(0)$. It follows from (19) that

$$\hat{H}_T\hat{\lambda}(0) = \hat{J}_T(\hat{x}(0) + \int_0^T e^{-\hat{A}t}\hat{B}u(t)\mathrm{d}t), \tag{20}$$

where $\hat{J}_T=Q_T^\Sigma e^{AT}$, $\hat{H}_T=[-Q_T^\Sigma,I_n]e^{\hat{A}T}[\mathbf{0},I_n]^\top$.

Proposition 2: Under Assumption 2, the optimal control $\hat{u}_i^*(t)$ of linear quardratic problem (1)-(18) is

$$-R_{ii}^{-1}B_i^\top e^{-A^\top t}\hat{H}_J(x(0)+\int_0^T e^{-At}Bu(t)\mathrm{d}t), \tag{21}$$

where $i=1,2$, $\hat{H}_J=(\hat{H}_T)^{-1}\hat{J}_T$.

Proof: The two-point boundary-value problem (19) can be converted into

$$(M_1 e^{-\hat{A}T}+M_2)w = [x_0^\top,\mathbf{0}]^\top, \tag{22}$$

where $w = e^{\hat{A}T}(\hat{x}(0) + \int_0^T e^{-\hat{A}t}\hat{B}u(t)\mathrm{d}t)$, and

$$M_1 = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, M_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -Q_T^\Sigma & I_n \end{bmatrix}.$$

Denote $[I_n, \mathbf{0}]e^{\hat{A}T} = [W_1, W_2]$, then for any $x(0) = x_0$, the solvability of (19) is equivalent to the solvability of (22) or

$$\begin{bmatrix} W_1 & W_2 \\ -Q_T^\Sigma & I_n \end{bmatrix} w = \begin{bmatrix} x_0 \\ \mathbf{0} \end{bmatrix}. \quad (23)$$

Left multiplying M for both sides of (23) gets

$$(W_1 + W_2 Q_T)x(T) = x(0),$$

where

$$M = \begin{bmatrix} I_n & -W_2 \\ \mathbf{0} & I_n \end{bmatrix}.$$

Due to $x(0) \in \mathbb{R}^n$, matrix $W_1 + W_2 Q_T$ is not singular, from which $\hat{x}(T) = [I_n, (Q_T^\Sigma)^\top]^\top (W_1 + W_2 Q_T)^{-1} x(0)$ can be implied. Combining to (20), we can see that $\hat{H}_T$ is a nonsingular matrix, since $\hat{H}_T \hat{\lambda}(0) = \mathbf{0}$ has a unique solution $\hat{\lambda}(0) = \mathbf{0}$ when $x(0) = \mathbf{0}$ and $u(t) \equiv \mathbf{0}$, $t \in [0, T]$. Now it is clear that $\hat{\lambda}(0) = \hat{H}_J(x(0) + \int_0^T e^{-At}Bu(t)\mathrm{d}t)$. Substituting it into (19) gets (21). ∎

From the above proposition, it can be concluded that under Assumption 2, the optimal control of linear quardratic problem (12)-(18) is

$$\hat{u}_i^*(t) = -R_{ii}^{-1} B_i^\top e^{-A^\top t} \hat{H}_J x(0), i = 1, 2,$$

which is denoted as $O_{kh}(x_0) := (\hat{u}_1^*, \hat{u}_2^*)$.

Definition 3: The regulation problem on Nash Equilibriums of (1)-(2) is solvable for $x(0) = x_0$ under the Kaldor-Hicks criterion, if there exists $u(t)$, such that $J(x_0, u(t), N_e(x_0, u(t))) < J(x_0, N_e(x_0))$ and $O_{kh}(x_0, u(t)) = N_e(x_0, u(t))$.

The regulation problem is said to be globally solvable under the Kaldor-Hicks criterion, if the regulation problem is solvable for any initial state x_0 under the Kaldor-Hicks criterion.

Then we discuss when the Nash equilibrium $N_e(x_0, u(t))$ of (1)-(2) is the optimal action profile $O_{kh}(x_0, u(t))$ of (1)-(18) under the Kaldor-Hicks criterion. Before this, we define a series of Gram matrices as follows:

$$W_{iT}^C = \int_0^T e^{-At} B_i B_i^\top (e^{-At})^\top \mathrm{d}t, i = 1, 2.$$

Proposition 3: Under Assumptions 1 and 2 and $|H_T| \neq 0$, for any $x_0 \in \mathbb{R}^n$, $O_{kh}(x_0, u(t)) = N_e(x_0, u(t))$ if and only if

$$\mathrm{Rank}(U, V) = \mathrm{Rank}(U). \quad (24)$$

where

$$U = VW_T^C, V = \begin{bmatrix} W_{1T}^C(\hat{H}_J - H_J^1) \\ W_{2T}^C(\hat{H}_J - H_J^2) \end{bmatrix}.$$

Proof: For GBCS (1)-(2), it follows from (14) that the Nash equilibrium generated by $x(0) = x$ and $u(t)$ is

$$u_i^*(t) = -R_{ii}^{-1} B_i^\top e^{-A^\top t} H_J^i (x(0) + \int_0^T e^{-At} Bu(t)\mathrm{d}t).$$

Combining the above equation with (21), (15), we can obtain that $O_{kh}(x_0, u(t)) = N_e(x_0, u(t))$, if and only if $\hat{u}_i^*(t) = u_i^*(t)$, $i = 1, 2$, $t \in [0, T]$, i.e., for $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$, such that

$$B_i^\top e^{-A^\top t}(H_J^i - \hat{H}_J)(x + W_T^C y) \equiv \mathbf{0}. \quad (25)$$

Left multiplying $A^{-t}B_i$ and integrating from 0 to T for both sides of the equality (25) gives that for any $x \in \mathbb{R}^n$, there exists $y \in \mathbb{R}^n$, such that $Vx = Uy$, which is equivalent to that matrix equation $UX = V$ has solutions, i,e., $\mathrm{Rank}(U, V) = \mathrm{Rank}(U)$.

Inverting the above procedure obtains that the regulation problem on Nash Equilibriums of (1)-(2) is solvable under the Kaldor-Hicks criterion, if $\mathrm{Rank}(U, V) = \mathrm{Rank}(U)$. ∎

From the above proposition, it can be concluded that under Assumptions 1 and 2, $|H_T| \neq 0$, if (A, B) is controllable, then $O_{kh}(x_0, u(t)) = N_e(x_0, u(t))$ for any $x_0 \in \mathbb{R}^n$.

Corollary 1: Under Assumptions 1 and 2 and $|H_T| \neq 0$, $O_{kh}(x_0) = N_e(x_0)$, if and only if

$$W_{iT}^C(\hat{H}_J - H_J^i) = \mathbf{0}, i = 1, 2. \quad (26)$$

Proof: If (13) is the optimal action profile of (12)-(18) under the Kaldor-Hicks criterion, then one can obtain $\hat{u}_i^*(t) \equiv u_i^*(t)$, $t \in [0, T]$, $i = 1, 2$. It follows from (13) and (19) that

$$B_i^\top e^{-A^\top t}(\hat{H}_J - H_J^i) = \mathbf{0}, \quad (27)$$

since R_{ii} is nonsingular. Left multiplying $A^{-t}B_i$ and integrating from 0 to T for both sides of the equality (27) gives $W_{iT}^C(\hat{H}_J - H_J^i) = \mathbf{0}$.

On the other hand, if $W_{iT}^C(\hat{H}_J - H_J^i) = \mathbf{0}$, then $(\hat{H}_J - H_J^i)^\top W_{iT}^C(\hat{H}_J - H_J^i) = \mathbf{0}$, i.e.,

$$\int_0^T (\hat{H}_J - H_J^i)^\top A^{-t} B_i B_i^\top (A^{-t})^\top (\hat{H}_J - H_J^i)\mathrm{d}t = \mathbf{0}, (28)$$

in which the integrand is positive semidefinite. By testifying that every column of $B_i^\top (A^{-t})^\top (\hat{H}_J - H_J^i)$, $t \in [0, T]$ is zero vector gets that (27) holds. Furthermore, combining (27) to (13) and (19) reveals that (13) is the optimal action profile of (12)-(18) under the Kaldor-Hicks criterion. ∎

Under Assumptions 1 and 2 and $|H_T| \neq 0$, (A, B_i), $i = 1, 2$, being controllable, $O_{kh}(x_0) = N_e(x_0)$, if and only if $H_J^1 = H_J^2 = \hat{H}_J$. Notice that the condition (26) can be equivalently expressed as $V = \mathbf{0}$, which is just a special case of Proposition 3. This means compared with the nominal system (12)-(18), it seems more possible that GBCS (1)-(2) generates the optimal action profile under Kaldor-Hicks criterion.

Proposition 4: Under Assumptions 1 and 2 and $|H_T| \neq 0$, the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable under the Kaldor-Hicks criterion, if and only if it is solvable for $x(0) = \mathbf{0}$ and (24) holds.

Proof: It suffices to show the sufficiency, since the necessity is trivial. By combining Definition 2 and Proposition 3, it can be found that the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable under the Kaldor-Hicks criterion, if and only if for any $x \in \mathbb{R}^n$, there exists y, such that

$$\begin{cases} Uy = Vx, \\ (x + W_T^C y)^\top Q(x + W_T^C y) < x^\top Qx. \end{cases} \quad \cdots \quad (29)$$

If is the regulation problem is solvable for $x(0) = \mathbf{0}$, then there exists $\bar{y} \in \mathbb{R}^n$, such that

$$\begin{cases} U\bar{y} = \mathbf{0}, \\ (W_T^C \bar{y})^\top Q(W_T^C \bar{y}) < 0. \end{cases} \quad \cdots \quad (30)$$

For any given $x(0) = x$, if (24) holds, i.e., $\mathrm{Rank}(U,V) = \mathrm{Rank}(U)$, then there exists $v_x \in \mathbb{R}^n$, such that $y = y_k := v_x + k \cdot \bar{y}$ satisfies $Uy = Vx$ for any $k \in \mathbb{R}$. Substituting $y = y_k$ into (29) gets

$$\begin{aligned} & (x + W_T^C y_k)^\top Q(x + W_T^C y_k) \\ = \; & (x + W_T^C v_x + k \cdot W_T^C \bar{y})^\top Q(x + W_T^C v_x + k \cdot W_T^C \bar{y}) \\ = \; & k^2 \cdot (W_T^C \bar{y})^\top Q(W_T^C \bar{y}) + 2k \cdot (x + W_T^C v_x)^\top Q W_T^C \bar{y} \\ & + v_x^\top W_T^C Q W_T^C v_x + 2x^\top Q W_T^C v_x + x^\top Qx, \end{aligned}$$

which is a quadratic function on k. Therefore we can find $K \in \mathbb{R}$, such that $(x + W_T^C y)^\top Q(x + W_T^C y) < x^\top Qx$, where $y = y_K$, since $(W_T^C \bar{y})^\top Q(W_T^C \bar{y}) < 0$. Combining the arbitrariness of $x(0)$ and (29) gives that under the Kaldor-Hicks criterion the regulation problem on Nash Equilibriums of (1)-(2) is globally solvable. ■

4. Conclusion

In GBCSs, the effectiveness of macro-regulation has been discussed under two criteria, which are Pareto and Kaldor-Hicks criterion, respectively. The former achieves the Pareto improvement on Nash Equilibriums, i.e., no one is harmed and at least one person is helped. The latter can achieve the potential Pareto improvement while generates optimal action profile, which is optimal from the holistic perspective to some extent, although every player only has the individual rather than collective rationality in this paper. Results revealed that the regulation of the third party helps to promote the consistency between individual and collective rationality under some conditions.

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References:

[1] T. Alpcan and T. Başar, "Network Security: A Decision and Game-Theoretic Approach", Cambridge University Press, 2010.

[2] R. Axelrod and W. D. Hamilton, "The evolution of cooperation", Science, 211(4489):1390–1396, 1981.

[3] T. Başar and G. J. Olsder, "Dynamic Noncooperative Game Theory", Society for Industrial and Applied Mathematics, 2nd edition edition, 1998.

[4] J. Engwerda, "LQ Dynamic Optimization and Differential Games", John Wiley & Sons, Ltd, 2005.

[5] E. Fehr and U. Fischbacher, "Social norms and human cooperation", Trends in Cognitive Sciences, 8(4):185–190, 2004.

[6] E. Fehr and U. Fischbacher, "Third-party punishment and social norms", Evolution and Human Behavior, 25(2):63–87, 2004.

[7] D. Fudenberg and J. Tirole, "Game Theory", MIT Press, 1991.

[8] R. Isaacs, "Differential games: A mathematical theory with applications to warfare and pursuit, control and optimization", 1965.

[9] D. Kennedy and C. Norman, "What Don't We Know?", Science, 309(5731):75–75, 2005.

[10] Y. Mu and L. Guo, "Towards a new paradigm of control theory", In Proceedings of the 1st China Automation Congress(CAC), pp. 26–49, 2009.

[11] Y. Mu and L. Guo, "How cooperation arises from rational players", Science China: Information Sciences, 56(11):1–9, 2013.

[12] Y. Mu and L. Guo, "Optimization and Identification in a Non-equilibrium Dynamic Game", In Proceedings of the 48th IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference, pp. 5750–5755, 2009.

[13] J. Nash, "Non-cooperative games", Annals of Mathematics, 54(2):286–295, 1951.

[14] J. V. Neumann and O. Morgenstern, "Theory of Games and Economic Behavior", Princeton University Press, 1944.

[15] M. A. Nowak, "Five rules for the evolution of cooperation", Science, 314(5805):1560–1563, 2006.

[16] E. Pennisi, "How did cooperative behavior evolve?", Science, 309(5731):93–93, 2005.

[17] W. Saad, Z. Han, H. V. Poor, and T. Başar, "Game-Theoretic Methods for the Smart Grid: An Overview of Microgrid Systems, Demand-Side Management, and Smart Grid Communications", IEEE Signal Processing Magazine, 29(5):86–105, 2012.

[18] P. Young and S. Zamir, "Handbook of Game Theory", Elsevier, 2014.

[19] Y. Yu and R. Zhang, "Controllability and Observability on Nash Equilibriums of Game-Based Control Systems", Submitted to IEEE Transactions on Automatic Control, 2021.

[20] R. Zhang and L. Guo, "Controllability of non-cooperative dynamical games", In Proceedings of the 34th Chinese Control Conference (CCC), pp. 353–358, 2015.

[21] R. Zhang and L. Guo, "Controllability of Nash equilibrium", In Proceedings of the 35th Chinese Control Conference (CCC), pp. 323–328, 2016.

[22] R. Zhang and L. Guo, "Controllability of Nash Equilibrium in Game-Based Control Systems", IEEE Transactions on Automatic Control, 64(10):4180–4187, 2019.

[23] R. Zhang and L. Guo, "Controllability of Stochastic Game-Based Control Systems", SIAM Journal on Control & Optimization, 57(6):3799–3826, 2019.