Paper:

Noise Reduction by Fuzzy Inference Based on α-Cuts and Generalized Mean: A Feasibility Study

Kiyohiko Uehara

Ibaraki University
Hitachi 316-8511, Japan
E-mail: kiyohiko.uehara.art@vc.ibaraki.ac.jp

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A method for noise reduction is proposed on the basis of a fuzzy inference method called α-GEMII. It is named α-GEMII-X denoising in this paper. α-GEMII-X denoising reduces noise in learning data by iteratively performing α-GEMII. α-GEMII-X denoising makes it possible to unify fuzzy inference and preprocessing to optimize fuzzy rules with noise-corrupted data. A method is also proposed for the determination of the timing when to terminate the iterative process by effectively using the properties of α-GEMII. Numerical results indicate that noise is properly reduced at the termination timing determined by the proposed method. They prove that α-GEMII-X denoising is feasible in practice.

Keywords: fuzzy inference, noise reduction, fuzzy rule optimization, α-cut, generalized mean

1. Introduction

Fuzzy inference has been applied to a wide variety of fields. It often uses a number of fuzzy rules in parallel to represent nonlinear systems. In applications, fuzzy rules are often optimized with learning data obtained by observation. In many cases, learning data are corrupted by noise. The deviation by noise is an obstacle to optimizing fuzzy rules. Fuzzy rules are required to be prevented from overfitting to noise in fuzzy rule optimization. Although a larger number of fuzzy rules can represent more complex nonlinear systems, an excessively large number of fuzzy rules often cause overfitting to noise. There is difficulty in making a balance between the representation ability of complex nonlinear systems and the avoidance of overfitting in optimizing the number of fuzzy rules. Therefore, noise reduction is significant in fuzzy rule optimization.

In this paper, a method is proposed for noise reduction based on α-GEMII (α-level-set and generalized-mean-based inference with the proof of two-sided symmetry of consequences) [1–4]. The proposed method is named *α-GEMII-X denoising* (α-GEMII-based denoising to unify fuzzy inference and preprocessing for fuzzy rule optimization). α-GEMII-X denoising effectively applies the mean-based operations in α-GEMII to noise reduction. It iteratively performs α-GEMII and reduces noise along with the iterations. Fuzzy rules for α-GEMII are initialized by directly using learning data. Facts are given by fuzzy sets directly generated from the learning data. The fuzzy rules are deterministically updated with deduced consequences in each iteration and noise is gradually reduced along with the iterations. A method is also proposed for the determination of the timing when to terminate the iterative process in α-GEMII-X denoising. They are derived by utilizing the changes of estimated deviations by noise in each iteration. The estimated deviations are stored in the form of the supports of consequent fuzzy sets defined in the fuzzy rules. Numerical results show the feasibility of α-GEMII-X denoising.

2. Definitions and Preliminaries

For the following discussions, some definitions and preliminaries are presented. Further details of each are described in [1–3].

Definition 1 When a convex fuzzy set A in the universe of discourse X is defined by a continuous membership function $\mu_A(x)\,(x \in X)$ and its α-cuts (also called α-level sets) are all bounded, *the reference point* x_A° of A is defined by using its α-cut A_α as follows:

$$x_A^\circ = \frac{x_\alpha^\ell + x_\alpha^u}{2}, \quad \alpha = \max_x \mu_A(x), \quad \ldots\ldots \quad (1)$$

where x_α^u and x_α^ℓ denote the least upper and the greatest lower bounds of A_α, respectively. ■

Definition 2 Suppose that a convex fuzzy set A in the universe of discourse X is defined by a continuous membership function $\mu_A(x)\,(x \in X)$ and its α-cuts are all bounded. The fuzzy set A is *symmetric* if and only if the following equation holds:

$$\frac{x_\alpha^\ell + x_\alpha^u}{2} = x_A^\circ, \quad \forall\,\alpha \in (0, \alpha_{\max}], \ \ \alpha_{\max} = \max_x \mu_A(x). \quad (2)$$

Here, x_α^u and x_α^ℓ denote the least upper and the greatest lower bounds of the α-cut A_α of A, respectively. The symbol x_A° represents the reference point of A. A convex fuzzy set is called *asymmetric* if and only if Eq. (2) does not hold. ■

Definition 3 The generalized mean $M(\{x_j, p_j\}; \omega)$ is defined by

$$M(\{x_j, p_j\};\omega) = \left[\frac{\sum_{j=1}^{n} p_j x_j^{\omega}}{\sum_{j=1}^{n} p_j}\right]^{\frac{1}{\omega}}, \quad x_j > 0, \quad p_j > 0, \quad (3)$$

where x_j denotes a real number in the universe of discourse and p_j represents a real number used for the weight of x_j. The symbol ω denotes a real number to determine the property of the mean [1, 2]. ■

3. α-GEMII and its Properties

The proposed method for noise reduction is based on α-GEMII [1–3]. This section introduces α-GEMII specialized for triangular membership functions.

3.1. Parallel Fuzzy Inference

This paper treats the parallel fuzzy inference in the form below:

$$\begin{array}{rl} \text{Rule 1:} & \text{If } x \text{ is } P_1 \text{ then } y \text{ is } Q_1. \\ \text{Rule 2:} & \text{If } x \text{ is } P_2 \text{ then } y \text{ is } Q_2. \\ \vdots & \qquad\qquad \vdots \\ \text{Rule } n\text{:} & \text{If } x \text{ is } P_n \text{ then } y \text{ is } Q_n. \\ \text{Given fact:} & x \text{ is } \tilde{P}. \\ \hline \text{Consequence:} & y \text{ is } \tilde{Q}. \end{array}$$

Here, P_j and $\tilde{P}$ denote fuzzy sets in the universe of discourse X, whereas Q_j and $\tilde{Q}$ represent fuzzy sets in the universe of discourse Y. In particular, P_j in the antecedent part of the fuzzy rule is called *an antecedent fuzzy set*, whereas Q_j in the consequent part of the fuzzy rule is called *a consequent fuzzy set*. When Q_j is defined by a singleton, it is especially called *a consequent singleton*. In this paper, P_j, Q_j, and $\tilde{P}$ are all defined by normal and convex fuzzy sets and their reference points are placed in $[0,1]$. The membership functions of P_j, Q_j, $\tilde{P}$, and $\tilde{Q}$ are respectively denoted by $\mu_{P_j}(x)$, $\mu_{Q_j}(y)$, $\mu_{\tilde{P}}(x)$, and $\mu_{\tilde{Q}}(y)$, where $x \in X$ and $y \in Y$. For convenience in the following discussions, the j-th fuzzy rule is represented by R_j.

3.2. α-GEMII Specialized for Triangular Membership Functions

Triangular membership functions are computationally effective in inference operations because the number of their parameters is small. Triangular membership functions can be parameterized with the least upper and the greatest lower bounds of α-cuts. As the operations in α-GEMII are α-cut-based, they can easily be specialized for triangular membership functions as shown below. The operational steps in the general form of α-GEMII are described in [1, 2].

Under the condition that P_j, Q_j, and $\tilde{P}$ are normal and are defined by triangular membership functions, α-GEMII can deduce consequences in the form of normal fuzzy sets defined by triangular membership functions. In the following discussion, note that the core of a fuzzy set is a singleton and is equal to its reference point when the fuzzy set is defined by a triangular membership function. Because a triangular membership function can be parameterized by its core and support, α-GEMII deduces only the core and support of the consequence by the effective use of its α-cut based scheme [1, 2]:

Deduction of cores: The core $y_{\tilde{Q}^c}$ of $\tilde{Q}$ is deduced by

$$y_{\tilde{Q}^c} = M(\{y_{Q_j^c}, \tilde{p}_j\};\omega(1)), \;\; \omega(1) = 1. \quad . \;\; . \;\; . \quad (4)$$

The core of Q_j is a singleton because Q_j is defined by a triangular membership function. In Eq. (4), $y_{Q_j^c}$ denotes the core of Q_j. Moreover, $\tilde{p}_j$ represents the compatibility degree between $\tilde{P}$ and P_j. The symbol $\omega(1)$ means the value of ω in the generalized mean for deducing cores. The value of $\omega(1)$ is set to 1 so that the deduced core is a singleton to make the membership functions of $\tilde{Q}$ triangular.

Deduction of supports: The least upper bound $y^u_{\underline{\tilde{Q}}}$ and the greatest lower bound $y^\ell_{\underline{\tilde{Q}}}$ of the support $\underline{\tilde{Q}}$ of $\tilde{Q}$ are deduced by

$$y^u_{\underline{\tilde{Q}}} = M(\{y^u_{\underline{Q}_j} + (1 - y_{\tilde{Q}^c}), \tilde{p}_j\};\; \omega(0)) - (1 - y_{\tilde{Q}^c}), (5)$$

$$y^\ell_{\underline{\tilde{Q}}} = \bar{M}(\{y^\ell_{\underline{Q}_j} - y_{\tilde{Q}^c}, \tilde{p}_j\};\; \omega(0)) + y_{\tilde{Q}^c}, \quad . \;\; . \;\; . \quad (6)$$

where $y^u_{\underline{Q}_j}$ and $y^\ell_{\underline{Q}_j}$ represent the least upper and the greatest lower bounds of the support $\underline{Q}_j$ of Q_j, respectively. The symbol $\omega(0)$ means the value of ω in the generalized mean for deducing supports. The function $\bar{M}(\{x_j, p_j\};\omega)$ is defined by

$$\bar{M}(\{x_j, p_j\};\omega) = 1 - M(\{1 - x_j,\; p_j\};\omega). \quad . \;\; . \;\; (7)$$

The operation with $\bar{M}$ is called *the dual operation of M* in this paper. The value of $\omega(0)$ determines the support of $\tilde{Q}$. In the general form of α-GEMII, the value of $\omega(0)$ is automatically controlled on the basis of the relation between $\tilde{P}$ and P_j $(j = 1, 2, \ldots, n)$ in fuzzy constraints. More details on the automatic control can be found in [2]. The scheme of α-GEMII for triangular membership functions is detailed in [1–4].

4. α-GEMII-X Denoising: Noise Reduction with α-GEMII

A method for noise reduction is proposed on the basis of α-GEMII. It iteratively performs α-GEMII to reduce noise. In this paper, the proposed method is named *α-GEMII-X denoising* (α-GEMII-based denoising to unify fuzzy inference and preprocessing for fuzzy rule optimization). A method is also proposed for the determination of the timing when to terminate the iterative process.

4.1. Operational Steps of α-GEMII-X Denoising

In α-GEMII-X denoising, α-GEMII is iteratively performed and noise in learning data is suppressed in each generation of the iterative process. In the following, the input-output pairs of the learning data are denoted by $(\hat{x}_k, \hat{y}_k)$ $(k = 1, 2, \ldots, n_d)$, where $\hat{x}_k$ and $\hat{y}_k$ are the input and output numerical values of the learning data, respectively. Suppose that the relations $\hat{x}_k < \hat{x}_{k+1}$ $(k = 1, 2, \ldots, n_d - 1)$ hold and the learning data are sampled at equal intervals. Although the membership-function shapes of the antecedent and consequent fuzzy sets determine the performance of noise reduction in α-GEMII-X denoising, triangular membership functions are adopted for their definitions in this paper. Triangular membership functions provide high computational efficiency in the operations of α-GEMII. As the antecedent and consequent fuzzy sets are defined by triangular membership functions, their cores are singletons. The operational steps of α-GEMII-X denoising are shown in the following:

Step 1: Initialize fuzzy rules for α-GEMII as stated below:

(i) Define the antecedent fuzzy sets P_k $(k = 1,2,\ldots,n_d)$ of the fuzzy rules R_k $(k = 1,2,\ldots,n_d)$ by triangular membership functions. Their cores $x^c_{P_k}$ $(k = 1,2,\ldots,n_d)$ are given by $\hat{x}_k$ $(k = 1,2,\ldots,n_d)$, respectively. The supports of P_k $(k = 1,2,\ldots,n_d)$ are set so as to form the strong fuzzy partitioning by satisfying

$$\sum_{k=1}^{n_d} \mu_{P_k}(x) = 1, \quad \ldots\ldots\ldots\ldots \quad (8)$$

where $\mu_{P_k}(x)$ denotes the membership function of P_k. Thus, P_k is symmetric and its support width is $2\Delta\hat{x}$, where $\Delta\hat{x} = \hat{x}_{k+1} - \hat{x}_k$ $(k = 1,2,\ldots,n_d - 1)$.

(ii) Define the consequent fuzzy sets Q_k $(k = 1,2,\ldots,n_d)$ of the fuzzy rules R_k $(k = 1,2,\ldots,n_d)$ by triangular membership functions. Their cores $y^c_{Q_k}$ $(k = 1,2,\ldots,n_d)$ are given by $\hat{y}_k$ $(k = 1,2,\ldots,n_d)$, respectively. The initial values of the least upper bound $y^u_{\underline{Q}_k}$ and the greatest lower bound $y^\ell_{\underline{Q}_k}$ of the support $\underline{Q}_k$ of Q_k are set to $\hat{y}_k$. Hence, Q_k is initially given by a singleton.

Step 2: Generate facts $\tilde{P}_k$ $(k = 1,2,\ldots,n_d)$ given for α-GEMII in the following way: $\tilde{P}_k$ $(k = 1,2,\ldots,n_d)$ are represented by symmetric fuzzy sets. The reference points of $\tilde{P}_k$ $(k = 1,2,\ldots,n_d)$ are set to $\hat{x}_k$ $(k = 1,2,\ldots,n_d)$, respectively. Each of their support-widths is less than or equal to $2\Delta x$. The membership function of $\tilde{P}_k$ is triangular or trapezoidal. It can also represent a closed interval because the membership function of a closed interval can be seen as a special case of trapezoidal membership functions. Note that the membership-function shape of $\tilde{P}_k$ determines the performance of noise reduction. Section 4.2 proposes a method for defining the membership function of $\tilde{P}_k$.

Step 3: In the first generation, give $\tilde{P}_k$ $(k = 1,2,\ldots,n_d)$ as facts and perform α-GEMII in accordance with R_k $(k = 1,2,\ldots,n_d)$. In other generations, give $\tilde{P}_k$ $(k = 2,\ldots,n_d - 1)$ as facts and perform α-GEMII in accordance with R_k $(k = 1,2,\ldots,n_d)$. Let $\tilde{Q}_k$ $(k = 1,2,\ldots,n_d)$ denote the consequences deduced with $\tilde{P}_k$ $(k = 1,2,\ldots,n_d)$, respectively. Note that the value of $\omega(0)$ is automatically set to 1 in the process of α-GEMII as stated in Section 3.2 [2].

Step 4: Replace the current values of $y^c_{Q_k}$ $(k = 1,2,\ldots,n_d)$ with the values of $y^c_{\tilde{Q}_k}$ $(k = 1,2,\ldots,n_d)$, respectively. The symbol $y^c_{\tilde{Q}_k}$ denotes the core of $\tilde{Q}_k$.

Step 5: Update the current values of $y^u_{\underline{Q}_k}$ and $y^\ell_{\underline{Q}_k}$ $(k = 1,2,\ldots,n_d)$ by using

$$y^u_{\underline{Q}_k} = y^c_{\tilde{Q}_k} + |y^c_{\tilde{Q}_k} - \hat{y}_k|, \quad (9)$$

$$y^\ell_{\underline{Q}_k} = y^c_{\tilde{Q}_k} - |y^c_{\tilde{Q}_k} - \hat{y}_k|. \quad (10)$$

Step 6: Finish the process if the termination conditions are satisfied; otherwise return to Step 3. The termination method is proposed in Section 4.3.

The deduced cores $y^c_{\tilde{Q}_k}$ $(k = 1,2,\ldots,n_d)$ present noise-reduced learning data. α-GEMII-X denoising effectively uses the mean operations in α-GEMII to reduce noise in each generation of the above-mentioned iterative process. The supports of Q_k $(k = 1,2,\ldots,n_d)$ indicate the possible ranges of the deviations caused by noise. In Section 4.3, criteria are proposed for the determination of the timing when to terminate the iterative process.

In Step 3, $\tilde{P}_1$ and $\tilde{P}_{n_d}$ are not given as facts after the first generation to prevent the deduced cores from moving to undesired directions in the iterative process. This is because each of $\tilde{P}_1$ and $\tilde{P}_{n_d}$ activates two fuzzy rules at the least upper and the greatest lower bounds of X. Adjacent three fuzzy rules are required to be activated to properly reduce noise in the iterative process as described in Section 4.2.

α-GEMII automatically sets the value of $\omega(0)$ to 1 in Step 3. Some variations are conceivable to reduce noise with α-GEMII: For example, the values of $\omega(1)$ and $\omega(0)$ are changed to other than 1, controlling by using some newly defined criteria. α-GEMII-X denoising can be further developed in the framework proposed in this paper.

In Step 3, α-GEMII deduces the singleton cores by setting the value of $\omega(1)$ to 1. These cores represent the noise-reduced learning data in α-GEMII-X denoising. When $\omega(1) = 1$, α-GEMII deduces the singleton cores in the equivalent manner of Takagi–Sugeno fuzzy inference in the case where the consequent parts of fuzzy rules are defined by constant values. In this sense, noise can be reduced also by using Takagi–Sugeno fuzzy inference only in the above-mentioned case and by applying the noise-reduction principle proposed in Section 4.2. Takagi–Sugeno fuzzy inference, however, cannot deduce fuzzy sets, as opposed to α-GEMII. Therefore, it cannot store the estimated deviations in the form of the supports of consequent fuzzy sets in the same manner as shown in Step 5. The transitional changes of the support widths are effectively utilized to determine the timing when to terminate the iterative process in α-GEMII-X denoising. Further details on the timing are described in Section 4.3.

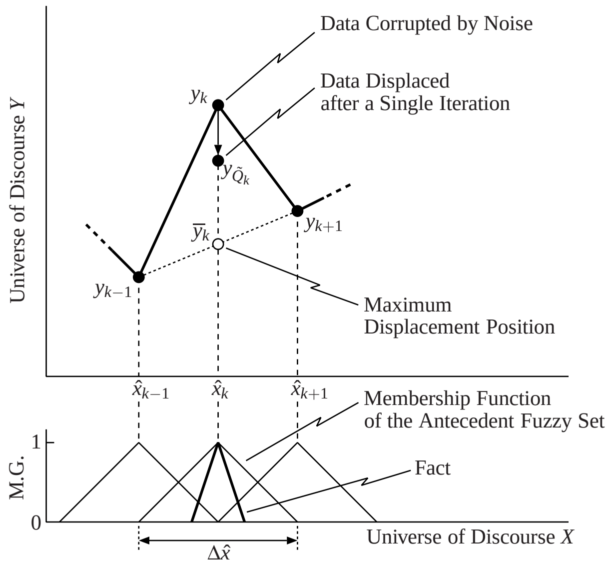


Fig. 1. Principle of noise reduction with α-GEMII-X denoising in the case where $y_k \geq \overline{y}_k$.

4.2. Principle of Noise Reduction with α-GEMII

In the following, let y_{k-1}, y_k, and y_{k+1} denote the current cores of Q_{k-1}, Q_k, and Q_{k+1} in a generation, respectively. In α-GEMII-X denoising, the compatibility degrees for P_{k-1}, P_k, and P_{k+1} are required to satisfy the conditions given by the following equations:

$$\tilde{p}_k = 1.0, \tag{11}$$

$$\tilde{p}_{k-1} = \tilde{p}_{k+1}. \tag{12}$$

Under the conditions, the deduced core $y^c_{\tilde{Q}_k}$ of $\tilde{Q}$ satisfies the relations below when the learning data are sampled at equal intervals:

$$\begin{cases} y_k \geq y^c_{\tilde{Q}_k} \geq \overline{y}_k, & y_k \geq \overline{y}_k, \\ y_k < y^c_{\tilde{Q}_k} < \overline{y}_k, & \text{otherwise}, \end{cases} \tag{13}$$

where $\overline{y}_k = (y_{k-1} + y_{k+1})/2$. The principle of noise reduction in α-GEMII-X denoising is based on Relation (13).

Figure 1 exemplifies the process of the noise reduction with α-GEMII-X denoising in the case where the relation $y_k \geq \overline{y}_k$ holds. Eq. (13) proves that the point $(\hat{x}_k, y_{\tilde{Q}_k})$ is always displaced from $(\hat{x}_k, y_k)$ toward $(\hat{x}_k, \overline{y}_k)$ in $X \times Y$ and can never exceed the point $(\hat{x}_k, \overline{y}_k)$ in each iteration. Since $\overline{y}_k$ is placed on the straight line between the points $(\hat{x}_{k-1}, y_{k-1})$ and $(\hat{x}_{k+1}, y_{k+1})$ in $X \times Y$, the noise-corrupted data are made locally closer to a linear sequence in $[\hat{x}_{k-1}, \hat{x}_{k+1}]$ in each iteration. Accordingly, the deviations due to noise in the learning data are globally suppressed. α-GEMII-X denoising reduces noise in such a deterministic way. The conditions with Eqs. (11) and (12) play an important role in the iterative process for noise reduction in α-GEMII-X denoising.

The above mentioned principle is derived by applying the essential mechanism of a noise reduction method called α-GEMI-ES (α-GEMINAS-based local-evolution toward slight linearity for global smoothness) [3]. α-GEMI-ES utilizes α-GEMINAS (α-level-set and generalized-mean-based inference with fuzzy rule interpolation at an infinite number of activating points) proposed for inference with sparse fuzzy rules [5]. In comparison with α-GEMI-ES, α-GEMII-X denoising is computationally cost-effective. Moreover, α-GEMII-X denoising is more flexible than α-GEMI-ES because it can control the noise-reduction performance and convergence speed by tuning the values of $\tilde{p}_{y_{k-1}}$ and $\tilde{p}_{y_{k+1}}$.

The process of the noise reduction in α-GEMII-X denoising is mathematically equivalent to the weighted moving average in the restricted way of weighting, wherein the weight restrictions are based on Eqs. (11) and (12). Thus, it can be seen that α-GEMII-X denoising iteratively performs the equivalent to the weighted moving average in the scheme of fuzzy inference with α-GEMII. Thereby, α-GEMII-X denoising can unify the fuzzy inference engine and the preprocessor for fuzzy rule optimization. It provides an effective way to implement fuzzy inference and noise reduction for fuzzy rule optimization on a single hardware platform.

4.3. A Method for Terminating Iterative Process in α-GEMII-X Denoising

In this Section, a method is proposed for the determination of the timing when to terminate the iterative process in α-GEMII-X denoising. It takes the effective use of the support widths of consequent fuzzy sets changed along with generations.

Let the change $\Delta W_{Q_k}(t)$ of the support width $W_{Q_k}(t)$ of $Q_k(t)$ be defined by $\Delta W_{Q_k}(t) = W_{Q_k}(t) - W_{Q_k}(t-1)$, where $Q_k(t)$ denotes the consequent fuzzy set in a generation t. The number of the fuzzy rules is counted in each generation, depending on $\Delta W_{Q_k}(t)$ as follows:

$$n_{\mathrm{inc}}(t): \ \Delta W_{Q_k}(t) > 0, \tag{14}$$

$$n_{\mathrm{non}}(t): \ \Delta W_{Q_k}(t) = 0, \tag{15}$$

$$n_{\mathrm{dec}}(t): \ \Delta W_{Q_k}(t) < 0. \tag{16}$$

If noise is ideally reduced along with the generations and the learning data are closer to their true values, the value of $\Delta W_{Q_k}(t)$ becomes smaller toward zero and hence the value of $n_{\mathrm{inc}}(t)$ decreases. If the iterative process in α-GEMII-X denoising is excessively performed, the learning data sequence is closer to a linear sequence even when the true sequence of the learning data is not originally linear. As a result, the value of $n_{\mathrm{inc}}(t)$ turns to increase when the iterative process is excessively conducted. By the effective use of this property of $n_{\mathrm{inc}}(t)$, this paper proposes a method for the determination of the timing when to terminate the iterative process in α-GEMII denoising.

As noise is not always reduced toward the true sequence in the iterative process, $n_{\mathrm{dec}}(t)$ is also considered together with $n_{\mathrm{inc}}(t)$. This paper proposes the use of the predominance of $n_{\mathrm{inc}}(t)$ over $n_{\mathrm{dec}}(t)$ as a criterion for deriving the timing when to terminate the iterative process in α-GEMII-X denoising. In order to numerically evaluate the predominance, the following index is proposed:

$$I_{\mathrm{incp}}(t) = [n_{\mathrm{inc}}(t) - n_{\mathrm{dec}}(t)]/n_d. \tag{17}$$

This index is named *the predominance index* I_{incp} in this paper. Using the predominance index as a criterion, this paper proposes a method for determining the termination timing of the iterative process in α-GEMII-X denoising as follows: The iterative process is terminated in around a generation where the value of $I_{\mathrm{incp}}(t)$ changes from decrease to increase.

The local linearity index I_{L_0} is also proposed to evaluate the smoothness of the noise-reduced data, considering the iterative process in α-GEMII-X denoising. The local linearity index I_{L_0} is defined for a sequence of singleton input–output pairs $(\hat{x}_k, y_k)$ $(k = 1, 2, ..., n)$ by using the following equation [3, 6]:

$$I_{L_0} = 1 - D, \quad (18)$$

where

$$D = \frac{1}{n-2} \sum_{k=2}^{n-1} d_k, \quad (19)$$

$$d_k = |y_k - \overline{y}_k|, \quad (20)$$

$$\overline{y}_k = s_k(\hat{x}_k), \quad (21)$$

$$s_k(x) = \frac{y_{k+1} - y_{k-1}}{\hat{x}_{k+1} - \hat{x}_{k-1}}(x - \hat{x}_{k-1}) + y_{k-1}. \quad (22)$$

Note that $s_k(x)$ is a linear function through the two points $(\hat{x}_{k-1}, y_{k-1})$ and $(\hat{x}_{k+1}, y_{k+1})$. Here, let $z_k(x)$ denote the piecewise linear function obtained by linearly connecting the three points $(\hat{x}_{k-1}, y_{k-1})$, $(\hat{x}_k, y_k)$, and $(\hat{x}_{k+1}, y_{k+1})$ $(k = 2, 3, \ldots, n-1)$. As can be found from the definition, I_{L_0} indicates the degree to which $z_k(x)$ is close to the linear function $s_k(x)$ locally in $[\hat{x}_{k-1}, \hat{x}_{k+1}]$. As the degree is higher, I_{L_0} is closer to 1. Thereby, the smoothness of a sequence is evaluated by the degree to which the sequence is locally closer to a linear sequence. The local linearity index I_{L_0} in a generation t is denoted by $I_{L_0}(t)$.

As criteria to terminate the iterative process in α-GEMII-X denoising, both $I_{\mathrm{incp}}(t)$ and $I_{L_0}(t)$ are effectively applied. Along with the generations, the value of $I_{L_0}(t)$ is closer to 1 as the smoothness of the noise-reduced sequence becomes higher, while $I_{\mathrm{incp}}(t)$ decreases and turns to increase as the noise-reduced sequence becomes closer to a linear sequence back away from the true sequence of the learning data as long as the true sequence of the learning data is not linear. Therefore, the iterative process in α-GEMII denoising is to be terminated at the timing when the value of $I_{L_0}(t)$ is large enough in around a generation where the value of $I_{\mathrm{incp}}(t)$ changes from decrease to increase.

5. Simulations: Transitional Changes in Noise Reduction with α-GEMII-X denoising

Simulations are performed in order to illustrate the basic properties of α-GEMII-X denoising. Their results indicate that it is feasible to apply α-GEMII-X denoising to fuzzy rule optimization with learning data corrupted by noise.

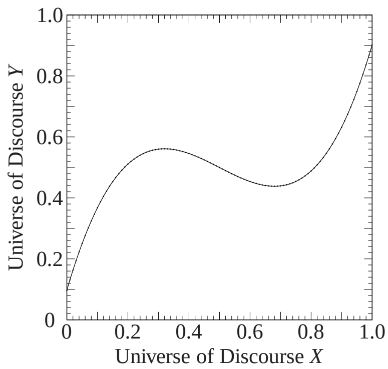


Fig. 2. Function $q(x)$

5.1. Simulation Conditions

The simulations are performed under the following conditions:

(i) Learning data are generated by the following equations:

$$\hat{y}_k = q(\hat{x}_k) + r_k, \quad k = 1, 2, \ldots, n_d, \quad (23)$$

$$q(x) = 0.12 \left[32 \left(\frac{x - 0.05}{0.9} \right)^3 - 48 \left(\frac{x - 0.05}{0.9} \right)^2 + 20.16 \left(\frac{x - 0.05}{0.9} \right) \right] + 0.25, \quad (24)$$

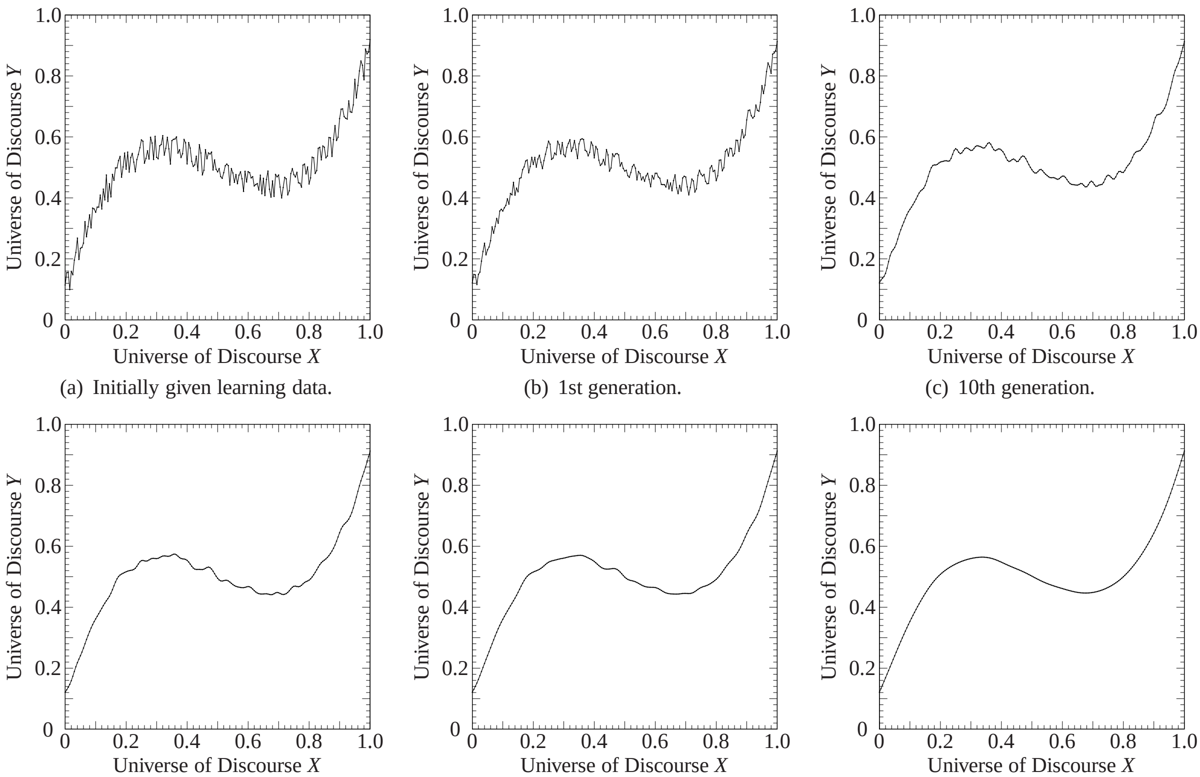
where r_k denotes the additional noise. The value of r_k is given by a uniform random number in $[-0.05, 0.05]$. The number n_d of the learning data is 201 and $\hat{x}_k = (k-1)/200$ $(k = 1, 2, \ldots, 201)$. Then, the value of $\Delta\hat{x}$ is 0.005. **Fig. 2** depicts the function $q(x)$.

(ii) The support width of each given fact $\tilde{P}_k$ is set to 0.25. This setting provides $\tilde{p}_{k-1} = 0.2$, $\tilde{p}_k = 1.0$, and $\tilde{p}_{k+1} = 0.2$. Therefore, Eqs. (11) and (12) hold.

5.2. Numerical Results

Figure 3 depicts the transitional changes in reducing noise in the learning data with α-GEMII-X denoising. The supports of the consequent fuzzy sets are not shown in the figure for ease of visibility. **Fig. 3(a)** shows the learning data which define the initial consequent cores $y^c_{Q_k}$ $(k = 1, 2, \ldots, 201)$ of the fuzzy rules in accordance with Step 1. **Figs. 3(b)–(f)** present the cores of the consequences deduced by α-GEMII after the first generation, followed by the 10th, 20th, 40th, and 200th generation of the iterative process in α-GEMII-X denoising. As can be found in these figures, the noise is reduced along with the generations.

Figures 4 and **5** show the transitional changes of I_{incp} and I_{L_0}, respectively. Considering both of them as stated in Section 4.3, the deduced cores in the 200th generation are adopted as final output of the process of α-GEMII-X denoising. As can be found from **Fig. 3(f)**, the smoothness and the closeness to the true sequence are well-balanced in the 200th generation.



(a) Initially given learning data. (b) 1st generation. (c) 10th generation.
(d) 20th generation. (e) 40th generation. (f) 200th generation.

Fig. 3. Transitional changes in noise reduction by α-GEMⅡ-X denoising.

From the discussions above, α-GEMⅡ-X denoising is found to be feasible. Further studies are to be conducted by using noise-corrupted data observed in the actual process in order to prove the practical effectiveness of α-GEMⅡ-X denoising.

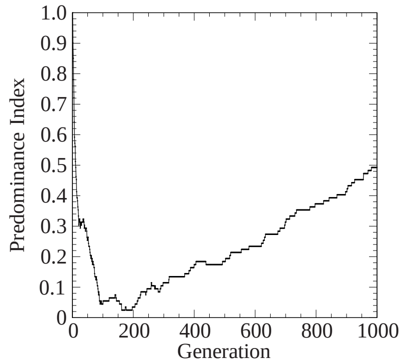


Fig. 4. Transitional changes in predominance index.

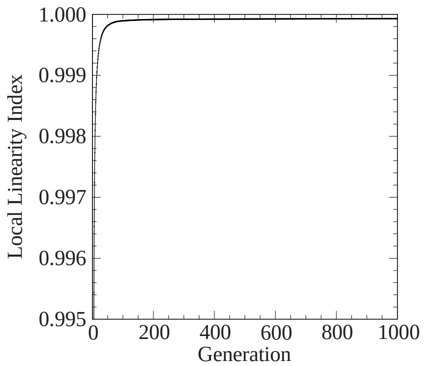


Fig. 5. Transitional changes in local linearity index.

6. Conclusion

A noise-reduction method has been proposed on the basis of a fuzzy inference method called α-GEMⅡ. The proposed method has been named *α-GEMⅡ-X denoising*. α-GEMⅡ-X denoising iteratively performs α-GEMⅡ to reduce noise in learning data sequences. A method has also been proposed for the determination of the timing when to terminate the iterative process. It can make a balance between the smoothness of the noise-reduced sequences and the closeness to the true sequences.

α-GEMⅡ-X denoising can unify fuzzy inference based on α-GEMⅡ and preprocessing to optimize fuzzy rules with noise-corrupted data. It provides a way to implement fuzzy inference and noise reduction for fuzzy rule optimization on a single hardware platform.

Numerical results have shown that α-GEMⅡ-X denoising properly reduces noise along with the iterations. The proposed method for terminating the iterative process in α-GEMⅡ-X denoising has been found to be effective: The smoothness and the closeness to the true sequence are well-balanced. Thereby, the numerical results indicate the feasibility of α-GEMⅡ-X denoising in practical use.

α-GEMII-X denoising may be further developed by the effective use of the properties of α-GEMII: the generalized mean operations and the proof of convex consequences. The development is expected to provide more effective methods for unifying fuzzy inference and fuzzy rule optimization in a single hardware platform.

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